

BLOW-UP EXAMPLES FOR SECOND ORDER ELLIPTIC PDES OF CRITICAL SOBOLEV GROWTH

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ABSTRACT. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and $\Delta_g = -\operatorname{div}_g \nabla$ be the Laplace-Beltrami operator. Let also 2^* be the critical Sobolev exponent for the embedding of the Sobolev space $H_1^2(M)$ into Lebesgue's spaces, and h be a smooth function on M . Elliptic equations of critical Sobolev growth such as

$$(E) \quad \Delta_g u + hu = u^{2^*-1}$$

have been the target of investigation for decades. A very nice H_1^2 -theory for the asymptotic behaviour of solutions of such equations has been available since the 1980's. The C^0 -theory was recently developed by Druet-Hebey-Robert. Such a theory provides sharp pointwise estimates for the asymptotic behaviour of solutions of (E) . It was used as a key point by Druet to prove compactness results for equations such as (E) . An important issue in the field of blow-up analysis, in particular with respect to previous work by Druet and Druet-Hebey-Robert, is to get explicit nontrivial examples of blowing-up sequences of solutions of (E) . We present such examples in this article.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. We denote by $H_1^2(M)$ the standard Sobolev space of functions in L^2 with one derivative in L^2 . We let (h_ε) be a sequence of $C^{0,\theta}$ functions on M , $0 < \theta < 1$, and consider equations such as

$$(E_\varepsilon) \quad \Delta_g u + h_\varepsilon u = u^{2^*-1},$$

where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace-Beltrami operator, $2^* = 2n/(n-2)$ is the critical Sobolev exponent for the embedding of the Sobolev space $H_1^2(M)$ into Lebesgue's spaces, and u is required to be positive. We let (u_ε) be a bounded sequence in $H_1^2(M)$ of solutions of (E_ε) in the sense that for any ε ,

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^*-1}$$

and $\|u_\varepsilon\|_{H_1^2} \leq \Lambda$ where $\Lambda > 0$ is independent of ε . We also assume that the h_ε 's are uniformly bounded and that they converge in L^2 to some limiting function h_0 . Then we know, since the 1980's, how to describe the H_1^2 -asymptotic behaviour of the u_ε 's as $\varepsilon \rightarrow 0$. Such a description when dealing with equations like (E_ε) was proved in Struwe [17]. Related references are Brézis-Coron [1, 2], Lions [12], Sacks-Uhlenbeck [14], Schoen [15], and Wente [18]. In particular, following [17], we do

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get that, up to a subsequence,

$$(0.1) \quad u_\varepsilon = u_0 + \sum_{i=1}^k B_\varepsilon^i + R_\varepsilon$$

and

$$(0.2) \quad \|u_\varepsilon\|_{H_1^2}^2 = \|u_0\|_{H_1^2}^2 + \sum_{i=1}^k \|B_\varepsilon^i\|_{H_1^2}^2 + r_\varepsilon,$$

where u_0 is a solution of the limit equation

$$(E_0) \quad \Delta_g u + h_0 u = u^{2^*-1},$$

where k is some nonnegative integer, $\|\cdot\|_{H_1^2}$ is the standard norm on $H_1^2(M)$, the $B_{i,\varepsilon}$'s are standard bubbles, $r_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, and $R_\varepsilon \rightarrow 0$ in $H_1^2(M)$ as $\varepsilon \rightarrow 0$.

Standard bubbles are rescalings of fundamental positive solutions of the Euclidean equation $\Delta u = u^{2^*-1}$. Such solutions have been classified by Caffarelli-Gidas-Spruck [3]. We refer also to Obata [13]. More precisely, a standard bubble is a sequence (B_ε) of functions given by

$$(0.3) \quad B_\varepsilon(x) = \left(\frac{\mu_\varepsilon}{\mu_\varepsilon^2 + \frac{d_g(x_\varepsilon, x)^2}{n(n-2)}} \right)^{\frac{n-2}{2}},$$

where (x_ε) is a converging sequence of points in M , and (μ_ε) is a sequence of positive real numbers such that $\mu_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. The x_ε 's are referred to as the centers of the bubble, and the μ_ε 's as the weights of the bubble. Clearly, $B_\varepsilon \rightarrow 0$ in $C_{loc}^0(M \setminus \{x_0\})$ as $\varepsilon \rightarrow 0$, where x_0 is the limit of the x_ε 's, while $B_\varepsilon(x_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Moreover, while $B_\varepsilon \rightarrow 0$ in L^2 as $\varepsilon \rightarrow 0$, we do have that

$$\|B_\varepsilon\|_{H_1^2}^2 = K_n^{-n} + r_\varepsilon$$

for all ε , where $r_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ and K_n , explicitly known and depending only on n , is the sharp constant K for the Euclidean Sobolev inequality $\|u\|_{2^*} \leq K \|\nabla u\|_2$. Independently, it is easily seen that a decomposition like (0.1)-(0.2) is unique in the following sense: (0.1)-(0.2) hold with respect to two sets (B_ε^i) and $(\tilde{B}_\varepsilon^i)$ of bubbles, $i = 1, \dots, k$, if and only if, up to renumbering,

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{\mu}_\varepsilon^i}{\mu_\varepsilon^i} = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{d_g(x_\varepsilon^i, \tilde{x}_\varepsilon^i)}{\mu_\varepsilon^i} = 0$$

for all $i = 1, \dots, k$, where the (B_ε^i) 's and $(\tilde{B}_\varepsilon^i)$'s are respectively defined with respect to (x_ε^i) , (μ_ε^i) and $(\tilde{x}_\varepsilon^i)$, $(\tilde{\mu}_\varepsilon^i)$. A possible reference of introductory nature on the subject is Hebey [9].

The decomposition (0.1)-(0.2) provides what we refer to as the H_1^2 -theory for blow-up. The C^0 -theory was recently developed in Druet-Hebey-Robert [6, 7]. It is proved in Druet-Hebey-Robert [7] that if the convergence of h_ε to h_0 is in $C^{0,\theta}(M)$, $\theta \in (0, 1)$, then, in addition to (0.1)-(0.2), there also exists $C > 1$ such that, up to

a subsequence,

$$\begin{aligned} & (1 - \eta_\varepsilon)u_0(x) + \frac{1}{C} \sum_{i=1}^k B_\varepsilon^i(x) \\ & \leq u_\varepsilon(x) \leq (1 + \eta_\varepsilon)u_0(x) + C \sum_{i=1}^k B_\varepsilon^i(x) \end{aligned}$$

for all ε and all x , where the η_ε 's are positive real numbers, independent of x , satisfying that $\eta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover (see again [6, 7]) the constant C can be chosen as close as we want to 1 in small neighbourhoods of the geometrical points, defined as the limits of the centers of the bubbles. In particular, the exact C^0 -asymptotic behaviour of the u_ε 's is obtained. This provides a complete description of the blow-up at the C^0 -level. Such a description was then used as a key point by Druet [4] when proving compactness results for equations such as (E_ε) in the spirit of those obtained by Schoen [15, 16]. As a remark, it is not possible to develop a general C^0 -theory for which R_ε in (0.1) would satisfy that $R_\varepsilon \rightarrow 0$ in $C^0(M)$ as $\varepsilon \rightarrow 0$.

Quite surprisingly, open questions, in particular with respect to the works developed in Druet [4] and Druet-Hebey-Robert [6, 7], are to know whether or not there exist examples of h_ε 's and u_ε 's such that in (0.1)-(0.2)

(Q1) $k \geq 1$, and $u_0(x) \neq 0$ for all x ,

or such that

(Q2) $k \geq 2$, and $d_g(x_\varepsilon^i, x_\varepsilon^j) \rightarrow 0$ as $\varepsilon \rightarrow 0$,

where the B_ε^i 's in the decomposition (0.1)-(0.2) are given by (0.3) with $x_\varepsilon = x_\varepsilon^i$ and $\mu_\varepsilon = \mu_\varepsilon^i$. As an interesting feature on (Q1), Druet [4] proved that if the convergence of h_ε to h_0 is in C^2 , and $n = 3, 4, 5$, then $u_0(x) = 0$ for all x . Theorem 0.1 below indicates that this is not anymore the case when $n \geq 6$. A positive answer to (Q2), as given by Theorem 0.1 below, shows that bubbles in the decomposition (0.1)-(0.2) associated to sequences of solutions of equations such as (E_ε) are not necessarily isolated. This result is false in dimension 3 where bubbles have to be isolated (see Li-Zhu [11]). Independently, thanks to Druet [4], we know that when $n \neq 6$, if the h_ε 's and u_ε 's are such that $k \geq 1$ in (0.1)-(0.2), and the convergence of h_ε to h_0 is in C^2 , then, necessarily, there exists at least one $x \in M$ such that

$$h_0(x) = \frac{n-2}{4(n-1)} S_g(x),$$

where S_g is the scalar curvature of g . A more precise statement is in Druet [4]. Assuming that the convergence of h_ε to h_0 does not hold in $C^2(M)$, we also address the question to know whether or not there exist examples of h_ε 's and u_ε 's such that

(Q3) $k \geq 1$, and $h_0(x) \neq \frac{n-2}{4(n-1)} S_g(x)$ for all x ,

where S_g is the scalar curvature of g . A more pertinent question is whether or not (especially when $n = 6$) we can ask that

(Q4) $k \geq 1$, $h_\varepsilon(x) \geq \frac{n-2}{4(n-1)} S_g(x) + \varepsilon_0$, and $h_0(x) \geq \frac{n-2}{4(n-1)} S_g(x) + \varepsilon_0$ for all x ,

where $\varepsilon_0 > 0$, and S_g is the scalar curvature of g .

We answer questions (Q1)-(Q4) in this article. The examples we provide involve easy mathematics. We let (S^n, h) be the unit n -sphere. The scalar curvature of h

is $S_h = n(n-1)$. We let Φ_0 be the constant function

$$\Phi_0 \equiv \frac{n(n-2)}{4}$$

and consider, as above, equations such as

$$(0.4) \quad \Delta_h u + h_\varepsilon u = u^{2^*-1}.$$

When $h_\varepsilon = \Phi_0$, (0.4) is the Yamabe equation on the unit n -sphere. If (u_ε) is a sequence of smooth positive solutions of (0.4) we then have that

$$(0.5) \quad \Delta_h u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^*-1}.$$

Assuming that the h_ε 's are bounded in $L^\infty(S^n)$, and that they converge in $L^2(S^n)$ to some continuous function h_0 as $\varepsilon \rightarrow 0$, decomposition (0.1)-(0.2) holds. We say that the u_ε 's blow up as $\varepsilon \rightarrow 0$ if $k \geq 1$ in (0.1)-(0.2). Our first result is as follows:

Theorem 0.1. *Let (S^n, h) be the unit n -sphere, $n \geq 6$. Then*

(T1) *there exist examples of h_ε 's and u_ε 's satisfying (0.5) such that the sequence (h_ε) is bounded in $L^\infty(S^n)$, the sequence (h_ε) converges to $h_0 = \Phi_0$ in $L^p(S^n)$ for all $p \geq 1$, the sequence (u_ε) blows up (so that $k \geq 1$), and $u_0(x) \neq 0$ for all x .*

(T2) *there exist examples of h_ε 's and u_ε 's satisfying (0.5) such that the sequence (h_ε) converges to $h_0 = \Phi_0$ in $C^1(S^n)$, the sequence (u_ε) blows up with two bubbles (so that $k \geq 2$), and the distances between the centers of the two bubbles go to 0 as $\varepsilon \rightarrow 0$.*

(T3) *there exist examples of h_ε 's and u_ε 's satisfying (0.5) such that the sequence (h_ε) is bounded in $L^\infty(S^n)$, the sequence (h_ε) converges to some continuous function h_0 in $L^p(S^n)$ for all $p \geq 1$, the sequence (u_ε) blows up (so that $k \geq 1$), and $h_0(x) \neq \Phi_0(x)$ for all x .*

When $n = 6$, there exist examples of h_ε 's and u_ε 's satisfying (0.5) such that the sequence (h_ε) is bounded in $L^\infty(S^n)$, the sequence (h_ε) converges to some continuous function h_0 in $L^p(S^n)$ for all $p \geq 1$, such that h_ε and h_0 satisfy that

$$h_\varepsilon \geq \Phi_0 + \varepsilon_0 \quad \text{and} \quad h_0 \geq \Phi_0 + \varepsilon_0$$

for all $\varepsilon > 0$ and some $\varepsilon_0 > 0$, and such that the sequence (u_ε) blows up (so that $k \geq 1$). In particular, when $n \geq 6$, questions (Q1)-(Q3) receive positive answers, and when $n = 6$, question (Q4) also receives a positive answer.

We know since the work of Schoen [15, 16] (see also Li-Zhang [10] and Li-Zhu [11]) that there is an a priori bound for the energy of solutions of the Yamabe equation. We define here the energy $E(u)$ of a function $u \in H_1^2$ by $E(u) = \|u\|_{2^*}^2$. Another question of interest, in particular with respect to the works developed in Druet [4] and Druet-Hebey-Robert [6, 7], is whether or not there exists such a bound for solutions of equations such as (E_ε) . The following result hopefully answers this question in the negative.

Theorem 0.2. *Let (S^n, h) be the unit n -sphere, $n \geq 6$. Then there exist examples of h_ε 's and u_ε 's satisfying (0.5) such that the sequence (h_ε) converges to $h_0 = \Phi_0$ in $C^1(S^n)$ and such that the energies $E(u_\varepsilon)$ of the u_ε 's satisfy that $E(u_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.*

An interesting feature in Theorem 0.2 is that the limit equation is the Yamabe equation for which, as already mentioned, we do have a priori bounds on the energy. In this specific case of the unit sphere, only one level is involved and the positive

solutions of the Yamabe equation are known. We prove Theorem 0.1 in section 1. Strong interactions are discussed at the end of the section. We prove Theorem 0.2 in section 2.

1. PROOF OF THEOREM 0.1

We prove Theorem 0.1 in what follows. We assume $n \geq 6$. Given $\varepsilon > 0$, and $i = 1, 2$, we let $(x_{i,\varepsilon}) \in S^n$ be two sequences of points in S^n , and $(\beta_{i,\varepsilon}) \in \mathbb{R}$ be two sequences of real numbers such that $\beta_{i,\varepsilon} > 1$ for all $i = 1, 2$ and all $\varepsilon > 0$. Given $\lambda > 0$, we let

$$u_\varepsilon = \lambda B_{1,\varepsilon} + B_{2,\varepsilon},$$

where

$$B_{i,\varepsilon}(x) = \left(\frac{n(n-2)}{4} (\beta_{i,\varepsilon}^2 - 1) \right)^{\frac{n-2}{4}} \left(\beta_{i,\varepsilon} - \cos d_h(x_{i,\varepsilon}, x) \right)^{1-\frac{n}{2}}.$$

For $i = 1, 2$, and any $\varepsilon > 0$,

$$\Delta_h B_{i,\varepsilon} + \frac{n(n-2)}{4} B_{i,\varepsilon} = B_{i,\varepsilon}^{2^*-1},$$

where $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Clearly, $u_\varepsilon \in C^\infty(S^n)$ and $u_\varepsilon > 0$ for all $\varepsilon > 0$. An easy claim is that

$$\Delta_h u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^*-1},$$

where the linear term h_ε is given by

$$h_\varepsilon = \Phi_0 + \Phi_\varepsilon$$

and

$$\Phi_\varepsilon = \frac{(\lambda B_{1,\varepsilon} + B_{2,\varepsilon})^{2^*-1} - \lambda B_{1,\varepsilon}^{2^*-1} - B_{2,\varepsilon}^{2^*-1}}{\lambda B_{1,\varepsilon} + B_{2,\varepsilon}}.$$

We assume that $n \geq 6$, and prove the following:

(P1) We fix $x_{1,\varepsilon} = x_{2,\varepsilon} = x_0$ in S^n , we fix $\beta_{1,\varepsilon} = \beta_1$, we fix $\lambda = 1$, and let $\beta_{2,\varepsilon} \rightarrow 1$. Then the sequence (h_ε) is bounded in $L^\infty(S^n)$, and the sequence (h_ε) converges to $h_0 = \Phi_0$ in $L^p(S^n)$ for all $p \geq 1$.

(P2) We fix $x_{1,\varepsilon} = x_{2,\varepsilon} = x_0$ in S^n , we fix $\beta_{1,\varepsilon} = \beta_1$, we fix $\lambda > 1$, and let $\beta_{2,\varepsilon} \rightarrow 1$. Then the sequence (h_ε) is bounded in $L^\infty(S^n)$, the sequence (h_ε) converges to some h_0 in $L^p(S^n)$ for all $p \geq 1$, with $h_0 > \Phi_0$. Moreover, when $n = 6$, there exists $\varepsilon_0 > 0$ such that $h_\varepsilon \geq \Phi_0 + \varepsilon_0$ for all ε .

(P3) We fix $x_{1,\varepsilon} = x_0$ in S^n , we let $x_{2,\varepsilon} = x_\varepsilon$ and $\beta_{1,\varepsilon} = \beta_{2,\varepsilon} = \beta_\varepsilon$ such that $\beta_\varepsilon \rightarrow 1$, $d_h(x_0, x_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and $d_h(x_0, x_\varepsilon) \gg (\beta_\varepsilon - 1)^{1/20}$ for all ε . We fix $\lambda = 1$. Then the sequence (h_ε) converges to $h_0 = \Phi_0$ in $C^1(S^n)$.

An easy claim is that (P1)-(P3) imply Theorem 0.1. We have indeed, as shown for instance in Druet-Hebey-Robert [7], that

$$B_{i,\varepsilon} = B_\varepsilon^i + R_{i,\varepsilon},$$

where B_ε^i is a bubble as in the decomposition (0.1), and $R_{i,\varepsilon} \rightarrow 0$ in H_1^2 as $\varepsilon \rightarrow 0$. If $B_{i,\varepsilon}$ is as above, and B_ε^i is as in (0.3), then $x_\varepsilon^i = x_{i,\varepsilon}$ and

$$\mu_\varepsilon^i = \sqrt{\frac{4(\beta_{i,\varepsilon} - 1)}{n(n-2)(\beta_{i,\varepsilon} + 1)}}.$$

We start with the proof of (P1)-(P2). Then we prove (P3).

Proof of (P1) and (P2). Let (y_ε) be a sequence of points in S^n . It is easily seen that the following propositions hold:

(1) If $B_{2,\varepsilon}(y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, then

$$\Phi_\varepsilon(y_\varepsilon) \rightarrow (\lambda^{2^*-2} - 1)B_1(y_0)^{2^*-2},$$

where $B_1 = B_{1,\varepsilon}$ and y_0 is the limit of the y_ε 's.

(2) If $B_{2,\varepsilon}(y_\varepsilon) \rightarrow C_0$ as $\varepsilon \rightarrow 0$, then $y_\varepsilon \rightarrow x_0$ and

$$\Phi_\varepsilon(y_\varepsilon) \rightarrow \frac{(\lambda B_1(x_0) + C_0)^{2^*-1} - \lambda B_1(x_0)^{2^*-1} - C_0^{2^*-1}}{\lambda B_1(x_0) + C_0},$$

where $B_1 = B_{1,\varepsilon}$.

(3) If $B_{2,\varepsilon}(y_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, then $y_\varepsilon \rightarrow x_0$ and we have that

$$\Phi_\varepsilon(y_\varepsilon) \rightarrow 0 \text{ if } n \geq 7 \text{ and } \Phi_\varepsilon(y_\varepsilon) \rightarrow 2\lambda B_1(x_0) \text{ if } n = 6,$$

where $B_1 = B_{1,\varepsilon}$.

This clearly proves that the h_ε 's are bounded in $L^\infty(S^n)$. It is then easily checked that

$$h_\varepsilon \rightarrow h_0 \text{ in } L^p(S^n)$$

for all $p \geq 1$ when $\varepsilon \rightarrow 0$, where

$$h_0 = \Phi_0 + (\lambda^{2^*-2} - 1)B_1^{2^*-2}$$

since $B_{2,\varepsilon} \rightarrow 0$ a.e. Moreover, it is clear that when $n = 6$,

$$\liminf_{\varepsilon \rightarrow 0} \inf_{S^n} (h_\varepsilon - \Phi_0) > 0$$

when $\lambda > 1$. This proves (P1)-(P2). \square

Now we prove (P3). For the sake of simplicity, we prove that C^0 -convergence holds in all dimensions $n \geq 6$, and restrict ourselves to the case $n = 6$ for the C^1 -convergence. Similar arguments to the ones developed below easily give that the C^1 -convergence also holds when $n \geq 7$.

Proof of (P3). We let (y_ε) be a sequence of points in S^n . We write that

$$\Phi_\varepsilon(y_\varepsilon) = \frac{(B_{1,\varepsilon}(y_\varepsilon) + B_{2,\varepsilon}(y_\varepsilon))^{2^*-1} - B_{1,\varepsilon}(y_\varepsilon)^{2^*-1} - B_{2,\varepsilon}(y_\varepsilon)^{2^*-1}}{B_{1,\varepsilon}(y_\varepsilon) + B_{2,\varepsilon}(y_\varepsilon)}.$$

We choose x_ε such that

$$(1.1) \quad d_h(x_\varepsilon, x_0) >> (\beta_\varepsilon - 1)^{\frac{1}{4}}.$$

We claim that

$$(1.2) \quad \liminf_{\varepsilon \rightarrow 0} B_{2,\varepsilon}(y_\varepsilon) > 0 \implies \lim_{\varepsilon \rightarrow 0} B_{1,\varepsilon}(y_\varepsilon) = 0$$

and prove the claim in what follows. Let us assume that $B_{1,\varepsilon}(y_\varepsilon) \geq C_0$ for some $C_0 > 0$ independent of ε . Then,

$$\Phi_0(\beta_\varepsilon^2 - 1) \geq C_0^{\frac{4}{n-2}} (\beta_\varepsilon - \cos d_h(x_0, y_\varepsilon))^2.$$

Since $\beta_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$, this clearly implies that $d_h(x_0, y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We then get that

$$\Phi_0(\beta_\varepsilon^2 - 1) \geq C_0^{\frac{4}{n-2}} \left(\beta_\varepsilon - 1 + \frac{d_h(x_0, y_\varepsilon)^2}{2} (1 + o(1)) \right)^2$$

so that

$$\beta_\varepsilon - 1 + \frac{d_h(x_0, y_\varepsilon)^2}{2} (1 + o(1)) = O\left(\sqrt{\beta_\varepsilon - 1}\right).$$

This clearly gives that

$$d_h(x_0, y_\varepsilon) = O\left((\beta_\varepsilon - 1)^{\frac{1}{4}}\right).$$

By (1.1), we then get that

$$d_h(x_\varepsilon, y_\varepsilon) = d_h(x_0, x_\varepsilon) (1 + o(1))$$

so that

$$\begin{aligned} B_{2,\varepsilon}(y_\varepsilon)^{\frac{4}{n-2}} &= O\left((\beta_\varepsilon - 1)(\beta_\varepsilon - \cos[d_h(x_0, x_\varepsilon)(1 + o(1))])^{-2}\right) \\ &= O\left(\frac{\beta_\varepsilon - 1}{\left(2(\beta_\varepsilon - 1) + d_h(x_0, x_\varepsilon)^2(1 + o(1))\right)^2}\right) \\ &= o(1) \end{aligned}$$

thanks to (1.1), a contradiction. This clearly proves the above claim.

Let us now estimate $\Phi_\varepsilon(y_\varepsilon)$. We distinguish two cases.

Case 1. We assume that $B_{2,\varepsilon}(y_\varepsilon) = o(B_{1,\varepsilon}(y_\varepsilon))$. Then

$$\Phi_\varepsilon(y_\varepsilon) = \frac{(2^* - 1)B_{1,\varepsilon}(y_\varepsilon)^{2^*-2}B_{2,\varepsilon}(y_\varepsilon)(1 + o(1)) - B_{2,\varepsilon}(y_\varepsilon)^{2^*-1}}{B_{1,\varepsilon}(y_\varepsilon)(1 + o(1))}.$$

By (1.2), it is clear that $B_{2,\varepsilon}(y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. If we also have that $B_{1,\varepsilon}(y_\varepsilon) \rightarrow 0$, we can write that

$$\Phi_\varepsilon(y_\varepsilon) = o\left(B_{1,\varepsilon}(y_\varepsilon)^{2^*-2}\right) = o(1).$$

If $B_{1,\varepsilon}(y_\varepsilon)$ does not converge to 0 as $\varepsilon \rightarrow 0$, then we get that

$$\Phi_\varepsilon(y_\varepsilon) = o(1) + o\left(B_{1,\varepsilon}(y_\varepsilon)^{2^*-3}\right) = o(1)$$

since $2^* \leq 3$ when $n \geq 6$. This proves that $\Phi_\varepsilon(y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. By symmetry, it is clear that this conclusion continues to hold when $B_{1,\varepsilon}(y_\varepsilon) = o(B_{2,\varepsilon}(y_\varepsilon))$.

Case 2. We assume that $\frac{B_{2,\varepsilon}(y_\varepsilon)}{B_{1,\varepsilon}(y_\varepsilon)} \rightarrow C_0$ as $\varepsilon \rightarrow 0$, for some $C_0 > 0$ real. Then

$$\Phi_\varepsilon(y_\varepsilon) = B_{1,\varepsilon}(y_\varepsilon)^{2^*-2} \left(\frac{(C_0 + 1)^{2^*-1} - C_0^{2^*-1} - 1}{C_0 + 1} + o(1) \right).$$

By (1.2), $B_{2,\varepsilon}(y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. It follows that $\Phi_\varepsilon(y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Thanks to cases 1 and 2, we get that $\|h_\varepsilon - \Phi_0\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$. We are now left with the proof that the convergence is C^1 . We restrict ourselves to the case $n = 6$. We choose x_ε such that

$$(1.3) \quad d_g(x_\varepsilon, x_0) \gg (\beta_\varepsilon - 1)^{\frac{n-2}{4(n-1)}}$$

and let (y_ε) be a sequence of points in S^n . We claim that

$$(1.4) \quad \liminf_{\varepsilon \rightarrow 0} |\nabla B_{2,\varepsilon}(y_\varepsilon)| > 0 \implies \lim_{\varepsilon \rightarrow 0} \frac{B_{2,\varepsilon}(y_\varepsilon)}{B_{1,\varepsilon}(y_\varepsilon)} = +\infty$$

and

$$(1.5) \quad \begin{aligned} & \liminf_{\varepsilon \rightarrow 0} |\nabla B_{2,\varepsilon}(y_\varepsilon)| > 0 \\ & \implies B_{1,\varepsilon}(y_\varepsilon) = (n(n-2))^{\frac{n-2}{4}} (\beta_\varepsilon^2 - 1)^{\frac{n-2}{4}} d_h(y_\varepsilon, x_0)^{2-n} (1 + o(1)) . \end{aligned}$$

We prove these claims. We assume that $|\nabla B_{2,\varepsilon}(y_\varepsilon)| \geq C_0$ for some $C_0 > 0$ independent of ε . Then

$$(\beta_\varepsilon - \cos d_h(x_\varepsilon, y_\varepsilon))^{\frac{n}{2}} = O\left((\beta_\varepsilon - 1)^{\frac{n-2}{4}} \sin d_h(x_\varepsilon, y_\varepsilon)\right) .$$

This implies that $d_h(x_\varepsilon, y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and thus that

$$\beta_\varepsilon - 1 + \frac{d_h(x_\varepsilon, y_\varepsilon)^2}{2} (1 + o(1)) = O\left((\beta_\varepsilon - 1)^{\frac{n-2}{2n}} d_h(x_\varepsilon, y_\varepsilon)^{\frac{2}{n}}\right) .$$

This clearly leads to

$$(1.6) \quad d_h(x_\varepsilon, y_\varepsilon) = O\left((\beta_\varepsilon - 1)^{\frac{n-2}{4(n-1)}}\right) .$$

By (1.3), we thus get that

$$d_h(x_0, y_\varepsilon) = d_h(x_0, x_\varepsilon) (1 + o(1))$$

which obviously proves (1.5). Thanks to (1.6) we can then write that

$$\begin{aligned} \frac{B_{2,\varepsilon}(y_\varepsilon)}{B_{1,\varepsilon}(y_\varepsilon)} &\geq 4^{-\frac{n-2}{4}} (\beta_\varepsilon - \cos d_h(x_\varepsilon, y_\varepsilon))^{1-\frac{n}{2}} d_h(x_0, x_\varepsilon)^{n-2} (1 + o(1)) \\ &\geq 4^{-\frac{n-2}{4}} \left(\beta_\varepsilon - 1 + \frac{d_h(x_\varepsilon, y_\varepsilon)^2}{2} (1 + o(1))\right)^{1-\frac{n}{2}} \\ &\quad \times d_h(x_0, x_\varepsilon)^{n-2} (1 + o(1)) \\ &\geq C (\beta_\varepsilon - 1)^{-\frac{(n-2)^2}{4(n-1)}} d_h(x_0, x_\varepsilon)^{n-2} (1 + o(1)) , \end{aligned}$$

where $C > 0$ is independent of ε . This proves (1.4) thanks to (1.3). We assume now that $n = 6$. Then

$$\nabla h_\varepsilon(y_\varepsilon) = 2 \frac{B_{2,\varepsilon}(y_\varepsilon)^2 \nabla B_{1,\varepsilon}(y_\varepsilon) + B_{1,\varepsilon}(y_\varepsilon)^2 \nabla B_{2,\varepsilon}(y_\varepsilon)}{(B_{1,\varepsilon}(y_\varepsilon) + B_{2,\varepsilon}(y_\varepsilon))^2} .$$

We distinguish two cases.

Case 1. We assume that $\frac{B_{2,\varepsilon}(y_\varepsilon)}{B_{1,\varepsilon}(y_\varepsilon)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then

$$|\nabla h_\varepsilon(y_\varepsilon)| = O\left(\frac{B_{2,\varepsilon}(y_\varepsilon)^2}{B_{1,\varepsilon}(y_\varepsilon)^2} |\nabla B_{1,\varepsilon}(y_\varepsilon)|\right) + O(|\nabla B_{2,\varepsilon}(y_\varepsilon)|) .$$

By (1.4), we have that $|\nabla B_{2,\varepsilon}(y_\varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$. As a starting point, we assume that $|\nabla B_{1,\varepsilon}(y_\varepsilon)| = O(1)$. Then we get that

$$|\nabla h_\varepsilon(y_\varepsilon)| = o(1) .$$

Assume now that $|\nabla B_{1,\varepsilon}(y_\varepsilon)| \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Then, thanks to the analogue of (1.5) for $B_{1,\varepsilon}$, noting that our assumption implies (see above) that

$$d_h(x_\varepsilon, y_\varepsilon) = d_h(x_0, x_\varepsilon) (1 + o(1)) ,$$

we get that

$$(1.7) \quad B_{2,\varepsilon}(y_\varepsilon) = O\left((\beta_\varepsilon - 1) d_h(x_\varepsilon, x_0)^{-4}\right) .$$

Since $|\nabla B_{1,\varepsilon}(y_\varepsilon)| \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, we have that $d_h(x_0, y_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ so that

$$\begin{aligned} \frac{|\nabla B_{1,\varepsilon}(y_\varepsilon)|}{B_{1,\varepsilon}(y_\varepsilon)^2} &= O\left(\frac{d_h(x_0, y_\varepsilon)}{\beta_\varepsilon - 1}(\beta_\varepsilon - 1 + \frac{d_h(x_0, y_\varepsilon)^2}{2}(1 + o(1)))\right) \\ &= O(d_h(x_0, y_\varepsilon)) + O\left(\frac{d_h(x_0, y_\varepsilon)^3}{\beta_\varepsilon - 1}\right). \end{aligned}$$

As in the proof of (1.4) and (1.5), we can prove that the analog of (1.6) for $B_{1,\varepsilon}$ holds. Namely we have that

$$d_h(x_0, y_\varepsilon) = O\left((\beta_\varepsilon - 1)^{\frac{1}{5}}\right).$$

Thus we get that

$$\frac{|\nabla B_{1,\varepsilon}(y_\varepsilon)|}{B_{1,\varepsilon}(y_\varepsilon)^2} = O\left((\beta_\varepsilon - 1)^{-\frac{2}{5}}\right).$$

Thanks to (1.7), we then get that

$$B_{2,\varepsilon}(y_\varepsilon) \frac{|\nabla B_{1,\varepsilon}(y_\varepsilon)|}{B_{1,\varepsilon}(y_\varepsilon)^2} = O\left((\beta_\varepsilon - 1)^{\frac{3}{5}} d_h(x_\varepsilon, x_0)^{-4}\right).$$

Choosing x_ε such that

$$d_h(x_\varepsilon, x_0) \gg (\beta_\varepsilon - 1)^{\frac{3}{20}}$$

we get that $|\nabla h_\varepsilon(y_\varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$. By symmetry, the conclusion holds also if we assume that $\frac{B_{1,\varepsilon}(y_\varepsilon)}{B_{2,\varepsilon}(y_\varepsilon)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Case 2. We assume that $\frac{B_{2,\varepsilon}(y_\varepsilon)}{B_{1,\varepsilon}(y_\varepsilon)} \rightarrow C_0$ as $\varepsilon \rightarrow 0$ for some $C_0 > 0$ real. Then

$$|\nabla h_\varepsilon(y_\varepsilon)| = O(|\nabla B_{1,\varepsilon}(y_\varepsilon)|) + O(|\nabla B_{2,\varepsilon}(y_\varepsilon)|).$$

Thanks to (1.4), and its analogue for $B_{1,\varepsilon}$, it is easily checked that $|\nabla B_{1,\varepsilon}(y_\varepsilon)| \rightarrow 0$ and $|\nabla B_{2,\varepsilon}(y_\varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus $|\nabla h_\varepsilon(y_\varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Thanks to these two cases, the convergence of the h_ε 's to Φ_0 is C^1 when $n = 6$. Similar arguments give that we also have that $h_\varepsilon \rightarrow \Phi_0$ in $C^1(S^n)$ when $n \geq 7$. This ends the proof of (P3). \square

The bubbles in (P3) interact in a weak sense at the C^0 -level. We define the strong interaction at the C^0 -level of two bubbles (B_ε) and (\tilde{B}_ε) by the existence of $C > 0$ and a sequence (\hat{x}_ε) of points in M such that

$$\liminf_{\varepsilon \rightarrow 0} B_\varepsilon(\hat{x}_\varepsilon) \geq C \quad \text{and} \quad \liminf_{\varepsilon \rightarrow 0} \tilde{B}_\varepsilon(\hat{x}_\varepsilon) \geq C.$$

An easy claim is that there exist examples of h_ε 's and u_ε 's satisfying (0.5) such that the sequence (h_ε) is bounded in $L^\infty(S^n)$, the sequence (h_ε) converges to $h_0 = \Phi_0$ in $L^p(S^n)$ for all $p \geq 1$, and the sequence $u_\varepsilon = B_{1,\varepsilon} + B_{2,\varepsilon}$ is such that $(B_{1,\varepsilon})$ and $(B_{2,\varepsilon})$ interact strongly at the C^0 -level. For the sake of simplicity, we prove this claim when $n = 6$. We let $\beta_{1,\varepsilon} = \beta_{2,\varepsilon} = \beta_\varepsilon$, where $\beta_\varepsilon > 1$ is such that $\beta_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$, and let $(x_{1,\varepsilon})$ and $(x_{2,\varepsilon})$ be such that $d_h(x_{1,\varepsilon}, x_{2,\varepsilon}) = 2(\beta_\varepsilon - 1)^{1/4}$ for all ε . As above, we let $B_{i,\varepsilon}$, $i = 1, 2$, be such that

$$B_{i,\varepsilon}(x) = 6(\beta_\varepsilon^2 - 1)(\beta_\varepsilon - \cos d_h(x_{i,\varepsilon}, x))^{-2}$$

and set $u_\varepsilon = B_{1,\varepsilon} + B_{2,\varepsilon}$. Then,

$$\Delta_h u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^*-1}$$

for all ε , where $h_\varepsilon = \Phi_0 + \Phi_\varepsilon$ and $\Phi_\varepsilon = 2B_{1,\varepsilon}B_{2,\varepsilon}/(B_{1,\varepsilon} + B_{2,\varepsilon})$. Let $\delta > 0$ be such that $\cos \theta \leq 1 - \frac{\theta^2}{4}$ if $|\theta| \leq \delta$. It is easily checked that $\Phi_\varepsilon \rightarrow 0$ in $C^0(S^n \setminus X_\varepsilon)$ as $\varepsilon \rightarrow 0$, where $X_\varepsilon = B_{x_{1,\varepsilon}}(\delta) \cap B_{x_{2,\varepsilon}}(\delta)$. Noting that

$$\begin{aligned} \beta_\varepsilon - \cos d_h(x_{i,\varepsilon}, x) &\geq \beta_\varepsilon - 1 + \frac{1}{4}\sqrt{\beta_\varepsilon - 1} \\ &\geq \frac{1}{8}\sqrt{\beta_\varepsilon - 1} \end{aligned}$$

if $(\beta_\varepsilon - 1)^{1/4} \leq d_h(x_{i,\varepsilon}, x) \leq \delta$, we also have that there exists $C > 0$ such that $\Phi_\varepsilon(x) \leq C$ for all $x \in \bigcup_{i=1,2} B_{x_{i,\varepsilon}}(\delta) \setminus B_{x_{i,\varepsilon}}((\beta_\varepsilon - 1)^{1/4})$. We have that $d_h(x_{1,\varepsilon}, x_{2,\varepsilon}) = 2(\beta_\varepsilon - 1)^{1/4}$. Hence,

$$B_{x_{1,\varepsilon}}((\beta_\varepsilon - 1)^{1/4}) \cap B_{x_{2,\varepsilon}}((\beta_\varepsilon - 1)^{1/4}) = \emptyset.$$

This proves that the Φ_ε 's are bounded in $L^\infty(S^n)$. Let $\delta_\varepsilon > 0$, $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, be such that $(\beta_\varepsilon - 1)^{1/8} \leq \delta_\varepsilon \leq \delta$. Then,

$$\beta_\varepsilon - \cos d_h(x_{i,\varepsilon}, x) \geq \frac{1}{8}(\beta_\varepsilon - 1)^{1/4}$$

if $d_h(x_{i,\varepsilon}, x) \geq \delta_\varepsilon$, so that $\Phi_\varepsilon \rightarrow 0$ in $C^0(S^n \setminus \hat{X}_\varepsilon)$ as $\varepsilon \rightarrow 0$, where \hat{X}_ε is given by $\hat{X}_\varepsilon = B_{x_{1,\varepsilon}}(\delta_\varepsilon) \cup B_{x_{2,\varepsilon}}(\delta_\varepsilon)$. In particular, since (Φ_ε) is bounded in $L^\infty(S^n)$, we get that for any $p \geq 1$, $\Phi_\varepsilon \rightarrow 0$ in $L^p(S^n)$ as $\varepsilon \rightarrow 0$. It follows that the above h_ε 's are bounded in $L^\infty(S^n)$ and that they converge to $h_0 = \Phi_0$ in $L^p(S^n)$ for all $p \geq 1$. Now we let x_ε be such that

$$d_h(x_{1,\varepsilon}, x_\varepsilon) = d_h(x_{2,\varepsilon}, x_\varepsilon) = (\beta_\varepsilon - 1)^{1/4}.$$

Then $B_{1,\varepsilon}(x_\varepsilon) = B_{2,\varepsilon}(x_\varepsilon)$, and we easily get that there exists $C > 0$ such that for $\varepsilon > 0$ sufficiently small,

$$\frac{\beta_\varepsilon^2 - 1}{(\beta_\varepsilon - \cos d_h(x_{i,\varepsilon}, x_\varepsilon))^2} \geq C$$

for all $i = 1, 2$. This proves that $(B_{1,\varepsilon})$ and $(B_{2,\varepsilon})$ interact strongly at the C^0 -level, and the above claim. Then it is easily checked that the representatives B_ε^i of the $B_{i,\varepsilon}$'s also interact strongly at the C^0 -level.

As a remark, several of the preceding results can be extended to more than one or two bubbles. For instance, if we assume $n = 6$, and let $u_\varepsilon = \sum B_{i,\varepsilon}$ where the $B_{i,\varepsilon}$'s are as above, then the u_ε 's satisfy (0.5) with

$$h_\varepsilon = \Phi_0 + 2 \frac{\sum_{i < j} B_{i,\varepsilon} B_{j,\varepsilon}}{\sum_i B_{i,\varepsilon}},$$

where $\Phi_0 \equiv \frac{n(n-2)}{4}$ is as above. When $n \neq 6$, and $u_\varepsilon = \sum B_{i,\varepsilon}$, we have that $h_\varepsilon = \Phi_0 + \Phi_\varepsilon$, where

$$\Phi_\varepsilon = \frac{(\sum B_{i,\varepsilon})^{2^*-1} - \sum B_{i,\varepsilon}^{2^*-1}}{\sum B_{i,\varepsilon}}$$

and $\Phi_0 \equiv \frac{n(n-2)}{4}$ is as above.

2. PROOF OF THEOREM 0.2

We prove Theorem 0.2 in what follows. Here again, for the sake of simplicity, we prove that C^0 -convergence holds in all dimensions $n \geq 6$, and restrict ourselves to the case $n = 6$ for the C^1 -convergence. Similar arguments to the ones developed below easily give the result when $n \geq 7$. We assume $n \geq 6$. We let (N_ε) be a sequence of integers such that $N_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Given $\varepsilon > 0$, we also let $x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}$ be N_ε points in M such that $x_{i,\varepsilon} \neq x_{j,\varepsilon}$ when $i \neq j$. We define

$$(2.1) \quad d_\varepsilon = \min_{i \neq j} d_h(x_{i,\varepsilon}, x_{j,\varepsilon}) .$$

Then we set

$$u_\varepsilon = \sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon},$$

where the $B_{i,\varepsilon}$'s are as in section 1, the $x_{i,\varepsilon}$'s are as above, the $\beta_{i,\varepsilon}$'s are such that $\beta_{i,\varepsilon} = \beta_\varepsilon$ for all i , and the β_ε 's are such that

$$(2.2) \quad \begin{aligned} & (\beta_\varepsilon - 1) N_\varepsilon^9 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \text{ and} \\ & \beta_\varepsilon - 1 \leq \frac{d_\varepsilon^8}{N_\varepsilon} . \end{aligned}$$

It is easily checked that $u_\varepsilon \in C^\infty(S^n)$, that $u_\varepsilon > 0$, and that

$$\Delta_h u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^*-1}$$

in S^n , where $h_\varepsilon = \Phi_0 + \Phi_\varepsilon$ and

$$\Phi_\varepsilon = \frac{\left(\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon} \right)^{2^*-1} - \sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon}^{2^*-1}}{\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon}} .$$

In particular, the u_ε 's and h_ε 's satisfy (0.5). First we prove that $\Phi_\varepsilon \rightarrow 0$ in $C^0(S^n)$ as $\varepsilon \rightarrow 0$, and thus that $h_\varepsilon \rightarrow \Phi_0$ in $C^0(S^n)$ as $\varepsilon \rightarrow 0$. Since $2^* - 1 \geq 1$, it is clear that $\Phi_\varepsilon \geq 0$ in S^n . We let $x_\varepsilon \in S^n$ be such that

$$\Phi_\varepsilon(x_\varepsilon) = \max_{S^n} \Phi_\varepsilon .$$

Up to renumbering, we may assume that

$$B_{1,\varepsilon}(x_\varepsilon) \geq B_{i,\varepsilon}(x_\varepsilon)$$

for all $i = 2, \dots, N_\varepsilon$. For $t \geq 0$, and $p = 2^* - 1$, $(1+t)^p \leq 1+pt+t^p$ since $2 \leq 2^* \leq 3$ when $n \geq 6$. Writing that

$$\begin{aligned} & B_{1,\varepsilon}(x_\varepsilon)^{2^*-2} \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \\ &= B_{1,\varepsilon}(x_\varepsilon) \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon)^{2^*-2} (B_{i,\varepsilon}(x_\varepsilon) B_{1,\varepsilon}(x_\varepsilon)^{-1})^{3-2^*} \\ &\leq B_{1,\varepsilon}(x_\varepsilon) \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon)^{2^*-2} \end{aligned}$$

we then get that

$$\begin{aligned}
& \left(\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right) \Phi_\varepsilon(x_\varepsilon) \\
& \leq \left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^{2^*-1} + (2^*-1) B_{1,\varepsilon}(x_\varepsilon)^{2^*-2} \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \\
& \leq \left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right) \left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^{2^*-2} + (2^*-1) B_{1,\varepsilon}(x_\varepsilon) \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon)^{2^*-2} \\
& \leq \left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right) \left(\left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^{2^*-2} + (2^*-1) \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon)^{2^*-2} \right)
\end{aligned}$$

and since $2^*-2 \leq 1$ when $n \geq 6$, we get in particular that

$$(2.3) \quad \Phi_\varepsilon(x_\varepsilon) \leq 2^* \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon)^{2^*-2}.$$

If $N_\varepsilon B_{1,\varepsilon}(x_\varepsilon)^{2^*-2} \rightarrow 0$ as $\varepsilon \rightarrow 0$, since $B_{1,\varepsilon}(x_\varepsilon) \geq B_{i,\varepsilon}(x_\varepsilon)$ for all $i = 2, \dots, N_\varepsilon$, we easily obtain that $\Phi_\varepsilon(x_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let us now assume that

$$N_\varepsilon B_{1,\varepsilon}(x_\varepsilon)^{2^*-2} \geq \delta_0$$

for some $\delta_0 > 0$. Then,

$$N_\varepsilon \frac{n(n-2)}{4} (\beta_\varepsilon^2 - 1) (\beta_\varepsilon - \cos d_h(x_{1,\varepsilon}, x_\varepsilon))^{-2} \geq \delta_0$$

and since $N_\varepsilon (\beta_\varepsilon - 1) \rightarrow 0$ (see (2.2)) we can write that

$$d_h(x_{1,\varepsilon}, x_\varepsilon) \leq (N_\varepsilon (\beta_\varepsilon - 1))^{\frac{1}{5}}$$

for $\varepsilon > 0$ small. By (2.2), $d_\varepsilon \geq (N_\varepsilon (\beta_\varepsilon - 1))^{\frac{1}{8}}$. It follows (see the definition (2.1) of d_ε) that

$$d_h(x_{i,\varepsilon}, x_\varepsilon) \geq (N_\varepsilon (\beta_\varepsilon - 1))^{\frac{1}{6}}$$

for all $i = 2, \dots, N_\varepsilon$, and $\varepsilon > 0$ small. In particular,

$$\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon)^{2^*-2} \leq \frac{N_\varepsilon n(n-2)(\beta_\varepsilon^2 - 1)}{4 \left(\beta_\varepsilon - \cos \left((N_\varepsilon (\beta_\varepsilon - 1))^{\frac{1}{6}} \right) \right)^2}$$

and we get that

$$(2.4) \quad \sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon)^{2^*-2} = O \left((N_\varepsilon (\beta_\varepsilon - 1))^{\frac{1}{3}} \right) = o(1).$$

By (2.3) and (2.4) we then obtain that $\Phi_\varepsilon(x_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. This proves that $\Phi_\varepsilon \rightarrow 0$ in $C^0(S^n)$ as $\varepsilon \rightarrow 0$, and thus that

$$h_\varepsilon \rightarrow \Phi_0 \text{ in } C^0(S^n)$$

as $\varepsilon \rightarrow 0$. Now we assume that $n = 6$, and prove that $\Phi_\varepsilon \rightarrow 0$ in $C^1(S^6)$ as $\varepsilon \rightarrow 0$, so that $h_\varepsilon \rightarrow \Phi_0$ in $C^1(S^6)$ as $\varepsilon \rightarrow 0$. We write that

$$\nabla \Phi_\varepsilon = \sum_{j=1}^{N_\varepsilon} \frac{\left(\sum_{i \neq j} B_{i,\varepsilon}^2 + \left(\sum_{i \neq j} B_{i,\varepsilon} \right)^2 \right) \nabla B_{j,\varepsilon}}{\left(\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon} \right)^2}$$

and then that

$$|\nabla \Phi_\varepsilon| \leq 2 \sum_{j=1}^{N_\varepsilon} \frac{\left(\sum_{i \neq j} B_{i,\varepsilon} \right)^2 |\nabla B_{j,\varepsilon}|}{\left(\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon} \right)^2}.$$

We let $x_\varepsilon \in S^6$ be such that

$$|\nabla \Phi_\varepsilon(x_\varepsilon)| = \max_{S^6} |\nabla \Phi_\varepsilon|.$$

Up to renumbering, we may assume that

$$\frac{\left(\sum_{i \neq j} B_{i,\varepsilon}(x_\varepsilon) \right)^2 |\nabla B_{j,\varepsilon}(x_\varepsilon)|}{\left(\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^2} \leq \frac{\left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^2 |\nabla B_{1,\varepsilon}(x_\varepsilon)|}{\left(\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^2}$$

for all $\varepsilon > 0$ and all $j = 2, \dots, N_\varepsilon$. Then,

$$(2.5) \quad \|\nabla \Phi_\varepsilon\|_{L^\infty(S^6)} \leq N_\varepsilon |\nabla B_{1,\varepsilon}(x_\varepsilon)| \frac{\left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^2}{\left(\sum_{i=1}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^2}.$$

If $N_\varepsilon |\nabla B_{1,\varepsilon}(x_\varepsilon)| \rightarrow 0$ as $\varepsilon \rightarrow 0$, we clearly get that $\|\nabla \Phi_\varepsilon\|_{L^\infty(S^6)} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let us now assume that

$$N_\varepsilon |\nabla B_{1,\varepsilon}(x_\varepsilon)| \geq \delta_0$$

for some $\delta_0 > 0$. Then

$$12N_\varepsilon (\beta_\varepsilon^2 - 1) \frac{\sin d_h(x_{1,\varepsilon}, x_\varepsilon)}{(\beta_\varepsilon - \cos d_h(x_{1,\varepsilon}, x_\varepsilon))^3} \geq \delta_0,$$

and since $N_\varepsilon(\beta_\varepsilon - 1) \rightarrow 0$ (see (2.2)) we can write that $d_h(x_{1,\varepsilon}, x_\varepsilon) \rightarrow 0$ and then that

$$(2.6) \quad d_h(x_{1,\varepsilon}, x_\varepsilon) = O\left((N_\varepsilon(\beta_\varepsilon - 1))^{\frac{1}{5}}\right).$$

By (2.2), $d_\varepsilon \geq (N_\varepsilon(\beta_\varepsilon - 1))^{\frac{1}{5}}$. It follows (see the definition (2.1) of d_ε) that

$$d_h(x_{i,\varepsilon}, x_\varepsilon) \geq (N_\varepsilon(\beta_\varepsilon - 1))^{\frac{1}{5}}$$

for all $i = 2, \dots, N_\varepsilon$, and $\varepsilon > 0$ small. Hence,

$$\left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^2 = O\left(N_\varepsilon^2 (\beta_\varepsilon - 1)^2 \left(\beta_\varepsilon - \cos\left((N_\varepsilon(\beta_\varepsilon - 1))^{\frac{1}{5}}\right) \right)^{-4}\right)$$

and we get that

$$(2.7) \quad \left(\sum_{i=2}^{N_\varepsilon} B_{i,\varepsilon}(x_\varepsilon) \right)^2 = O\left(N_\varepsilon^{\frac{2}{3}} (\beta_\varepsilon - 1)^{\frac{2}{3}}\right).$$

Independently,

$$\begin{aligned} \frac{|\nabla B_{1,\varepsilon}(x_\varepsilon)|}{B_{1,\varepsilon}(x_\varepsilon)^2} &= O\left(\frac{d_h(x_{1,\varepsilon}, x_\varepsilon)}{\beta_\varepsilon - 1}(\beta_\varepsilon - \cos d_h(x_{1,\varepsilon}, x_\varepsilon))\right) \\ &= O(d_h(x_{1,\varepsilon}, x_\varepsilon)) + O\left(\frac{d_h(x_{1,\varepsilon}, x_\varepsilon)^3}{\beta_\varepsilon - 1}\right). \end{aligned}$$

Thanks to (2.6), this implies that

$$(2.8) \quad \frac{|\nabla B_{1,\varepsilon}(x_\varepsilon)|}{B_{1,\varepsilon}(x_\varepsilon)^2} = O\left(N_\varepsilon^{\frac{3}{5}}(\beta_\varepsilon - 1)^{-\frac{2}{5}}\right).$$

Coming back to (2.5), thanks to (2.7) and (2.8), we then get that

$$\|\nabla \Phi_\varepsilon\|_{L^\infty(S^6)} = O\left(N_\varepsilon^{\frac{34}{5}}(\beta_\varepsilon - 1)^{\frac{4}{15}}\right) = o(1)$$

thanks to (2.2). This proves that $\Phi_\varepsilon \rightarrow 0$ in $C^1(S^6)$ as $\varepsilon \rightarrow 0$, and thus that

$$h_\varepsilon \rightarrow \Phi_0 \text{ in } C^1(S^6)$$

as $\varepsilon \rightarrow 0$. Similar arguments give that $\Phi_\varepsilon \rightarrow 0$ in $C^1(S^n)$ when $n \geq 7$, and hence that $h_\varepsilon \rightarrow \Phi_0$ in $C^1(S^n)$ when $n \geq 7$. Now, in order to end the proof of Theorem 0.2, there we still need to prove that the energy $E(u_\varepsilon)$ of u_ε goes to $+\infty$ as $\varepsilon \rightarrow 0$. Noting that

$$\int_{S^n} u_\varepsilon^{2^*} dv_h \geq \sum_{i=1}^{N_\varepsilon} \int_{S^n} B_{i,\varepsilon}^{2^*} dv_h$$

and (see for instance Hebey [8]) that

$$\int_{S^n} B_{i,\varepsilon}^{2^*} dv_h = \left(\frac{n(n-2)}{4}\right)^{n/2} \omega_n$$

for all i and all $\varepsilon > 0$, where $\omega_n > 0$ is the volume of (S^n, h) , we get that

$$E(u_\varepsilon) \geq N_\varepsilon \left(\frac{n(n-2)}{4}\right)^{n/2} \omega_n.$$

In particular, $E(u_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, and this ends the proof of Theorem 0.2.

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